Wireless Network Pricing
Chapter 4: Social Optimal Pricing

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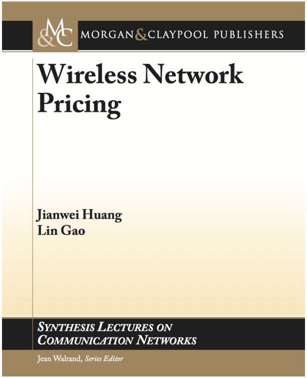
The Book



- E-Book freely downloadable from NCEL website: `http://ncel.ie.cuhk.edu.hk/content/wireless-network-pricing`
- Physical book available for purchase from Morgan & Claypool (`http://goo.gl/JFGlai`) and Amazon (`http://goo.gl/JQKaEq`)

Chapter 4: Social Optimal Pricing

Focus of This Chapter

- Key Focus: This chapter focuses on the issue of social optimal pricing, where one service provider chooses prices to maximize the social welfare.
- Theoretic Approach: Convex Optimization

Convex Optimization

- Largely follow the discussions in book "Convex Optimization" by Stephen Boyd and Lieven Vandenberghe.

Definition (Convex Optimization)

Convex optimization studies the problem of minimizing convex functions (or equivalently, maximizing concave functions) over convex sets.

Section 4.1
Theory: Dual-based Optimization

Prelims

- Notations
  - $\mathbb{R}^n$: the set of all real n-vectors
    - Each vector in $\mathbb{R}^n$ is called a *point* of $\mathbb{R}^n$.
    - $\mathbb{R}^1$ or $\mathbb{R}$ denotes the set of all real 1-vectors or all real numbers.
  - $\mathbb{R}^{m \times n}$: the set of all $m \times n$ real matrices
  - $f : \mathbb{R}^n \to \mathbb{R}^m$: a function that maps some real n-vectors (called the *domain* of function f) into real m-vectors
    - $\mathcal{D}(f)$: the domain of function f
- Concepts
  - Convex Set
  - Convex Function

Convex Set

Definition (Convex Set)

A nonempty set $\mathcal{X} \subseteq \mathbb{R}^n$ is convex, if for any $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathcal{X}$ and any $\theta \in \mathbb{R}$ with $0 \leq \theta \leq 1$, we have:

$$\theta \boldsymbol{x}_1 + (1-\theta)\boldsymbol{x}_2 \in \mathcal{X}$$

Convex Set

- Geometrically, a set is convex if every point in the set can be reached by every other point, along an inner straight path between them.
- Examples of convex and non-convex sets:

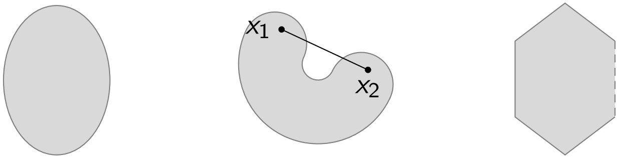


Figure: (i) Convex, (ii) Non-convex, and (iii) Non-convex.

Convex Combinition

- Convex Combination: A convex combination of points $\boldsymbol{x}_1, ..., \boldsymbol{x}_k$ can be expressed as

$$\boldsymbol{y} = \theta_1 \boldsymbol{x}_1 + ... + \theta_k \boldsymbol{x}_k,$$

with $\theta_1 + ... + \theta_k = 1$ and $\theta_i \geq 0, i = 1, ..., k$.

Lemma (4.2)

A nonempty set $\mathcal{X}$ is convex, if and only if the convex combination of any points in $\mathcal{X}$ also lies in $\mathcal{X}$.

Convex Hull

- Convex Hull: The convex hull of a set $\mathcal{X}$, denoted $\mathcal{H}(\mathcal{X})$, is the smallest convex set that contains $\mathcal{X}$.

Definition (Convex Hull)

The convex hull $\mathcal{H}(\mathcal{X})$ of a set $\mathcal{X}$ consists of the convex combinations of all points in $\mathcal{X}$, i.e.,

$$\{\theta_1 \boldsymbol{x}_1 + ... + \theta_k \boldsymbol{x}_k \mid \theta_1 + ... + \theta_k = 1, \theta_i \geq 0, \boldsymbol{x}_i \in \mathcal{X}, i = 1, ..., k\}.$$

- Properties
  - $\mathcal{H}(\mathcal{X})$ is always convex;
  - $\mathcal{X} \subseteq \mathcal{H}(\mathcal{X})$;
  - $\mathcal{X} = \mathcal{H}(\mathcal{X})$ if $\mathcal{X}$ is a convex set;
  - $\mathcal{H}(\mathcal{X}) \subseteq \mathcal{Y}$ where $\mathcal{Y}$ is any convex set that contains $\mathcal{X}$.

Convex Hull

- Examples of convex hull
  - Source sets:

  - Convex hulls:

Operations Preserving Convexity of Sets

- Intersection: Suppose $\mathcal{X}_1, ..., \mathcal{X}_k$ are convex sets. Then, the intersection of $\mathcal{X}_1, ..., \mathcal{X}_k$

$$\mathcal{X} \triangleq \mathcal{X}_1 \cap ... \cap \mathcal{X}_k$$

  is also a convex set.

- Affine Mapping: Suppose $\mathcal{X}$ is a convex set in $\mathbb{R}^n$, $\mathbf{A} \in \mathbb{R}^{m\times n}$, and $\boldsymbol{b} \in \mathbb{R}^m$. Then, the affine mapping of $\mathcal{X}$

$$\mathcal{Y} \triangleq \{\mathbf{A}\boldsymbol{x} + \boldsymbol{b} \mid \boldsymbol{x} \in \mathcal{X}\}$$

  is also a convex set.

Convex (and Concave) Function

Definition (Convex Function)

A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex, if

1. $\mathcal{D}(f)$ is a convex set, and
2. for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{D}(f)$ and $\theta \in \mathbb{R}$ with $0 \leq \theta \leq 1$, we have:

$$f(\theta \boldsymbol{x} + (1-\theta)\boldsymbol{y}) \leq \theta f(\boldsymbol{x}) + (1-\theta) f(\boldsymbol{y})$$

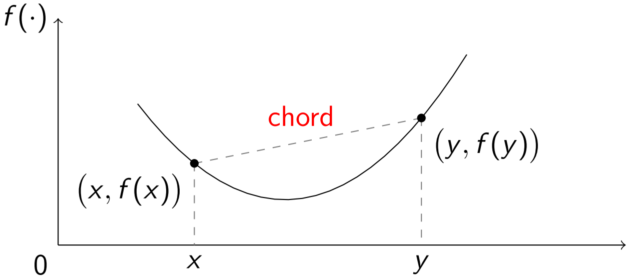
Definition (Concave Function)

A function $f(\cdot)$ is concave if and only if $-f(\cdot)$ is convex.

- A function $f(\cdot)$ can be neither convex nor concave, e.g., $f(x) = x^3$.

Convex Function

- Geometrically, a function $f(\cdot)$ is convex if the chord from any point $(\boldsymbol{x}, f(\boldsymbol{x}))$ to $(\boldsymbol{y}, f(\boldsymbol{y}))$ lies above the graph of $f(\cdot)$.
- Illustration of Convex Function $f(\cdot)$:



Generalized Definition of Convex Function

Definition (Convex Function)

A function $f(\cdot)$ is convex, if and only if (i) $\mathcal{D}(f)$ is convex and (ii)

$$f(\theta_1 \boldsymbol{x}_1 + ... + \theta_k \boldsymbol{x}_k) \leq \theta_1 f(\boldsymbol{x}_1) + ... + \theta_k f(\boldsymbol{x}_k),$$

for any $\boldsymbol{x}_1, ..., \boldsymbol{x}_k \in \mathcal{D}(f)$, when $\theta_1 + ... + \theta_k = 1$ and $\theta_i \geq 0, i = 1, ..., k$.

- Examples of convex functions
  - 2^x, 3^x, e^x, etc.
  - x^2, x^4, x^6, etc.
  - $-log_2(x)$, $-ln(x)$, etc.
  - ...

First-Order Condition

- First-Order Derivative (Gradient): the first-order derivative of a scalar-valued function $f(\cdot)$ at a point $\boldsymbol{x} \in \mathcal{D}(f)$, denoted by $\nabla f(\boldsymbol{x})$, is an n-vector with the i-th component given by

$$\nabla f(\boldsymbol{x})_i = \frac{\partial f(\boldsymbol{x})}{\partial x_i}, \ \ i = 1, ..., n,$$

  - x_i: the i-th coordinate of the vector $\boldsymbol{x}$;
  - $\frac{\partial f(\boldsymbol{x})}{\partial x_i}$: the partial derivative of $f(\boldsymbol{x})$ with respect to x_i.

Lemma (First-Order Condition)

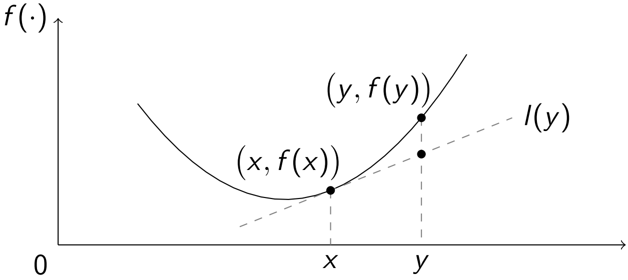
A differentiable function $f(\cdot)$ is convex, if and only if $\mathcal{D}(f)$ is convex and

$$f(\boldsymbol{y}) \geq f(\boldsymbol{x}) + \nabla f(\boldsymbol{x})^T(\boldsymbol{y} - \boldsymbol{x}), \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \mathcal{D}(f).$$

First-Order Condition

- Geometrically, the first-order condition means that the line passing through any point $(\boldsymbol{x}, f(\boldsymbol{x}))$ along the gradient direction $\nabla f(\boldsymbol{x})$ lies under the graph of $f(\cdot)$.

- Illustration of First-order Condition:



Second-Order Condition

- Second-Order Derivative (Hessian Matrix): the second-order derivative of a scalar-valued function $f(\cdot)$ at a point $\boldsymbol{x} \in \mathcal{D}(f)$, denoted by $\nabla^2 f(\boldsymbol{x})$, is an $n \times n$ matrix, given by

$$\nabla^2 f(\boldsymbol{x})_{ij} = \frac{\partial^2 f(\boldsymbol{x})}{\partial x_i \partial x_j}, \; i = 1, ..., n, j = 1, ..., n.$$

  - $\frac{\partial^2 f(\boldsymbol{x})}{\partial x_i \partial x_j}$: the second partial derivative of $f(\boldsymbol{x})$ with respect to x_i and x_j.

Lemma (Second-Order Condition)

A twice differentiable function $f(\cdot)$ is convex, if and only if $\mathcal{D}(f)$ is convex and its Hessian matrix is positive semidefinite, i.e.,

$$\nabla^2 f(\boldsymbol{x}) \succeq 0, \quad \forall \boldsymbol{x} \in \mathcal{D}(f).$$

Convex Function

- Operations Preserving Convexity of Functions
  - Nonnegative weighted sums: Suppose $f_1(\cdot), ..., f_k(\cdot)$ are convex, and $\theta_1, ..., \theta_k \geq 0$. Then the following function is convex:

$$f(\boldsymbol{x}) \triangleq \theta_1 f_1(\boldsymbol{x}) + ... + \theta_k f_k(\boldsymbol{x})$$

  - Composition with an affine mapping: Suppose $g(\cdot)$ is a convex function on $\mathbb{R}^n$, $\mathbf{A} \in \mathbb{R}^{n \times m}$, and $\boldsymbol{b} \in \mathbb{R}^n$. Then the following function is convex:

$$f(\boldsymbol{x}) \triangleq g(\mathbf{A}\boldsymbol{x} + \boldsymbol{b})$$

  - Point-wise maximum: Suppose $f_1(\cdot), ..., f_k(\cdot)$ are convex. Then the following function is convex:

$$f(\boldsymbol{x}) \triangleq \max\{f_1(\boldsymbol{x}), ..., f_k(\boldsymbol{x})\}$$

Convex Optimization

- Optimization Problem: the problem of finding a point $\boldsymbol{x}$ over a feasible set that minimizes an objective function:

Optimization Problem

$$\begin{aligned} \text{minimize} \quad & f(\boldsymbol{x}) \\ \text{subject to} \quad & f_i(\boldsymbol{x}) \leq 0, \ \ i = 1, ..., m. \end{aligned}$$

  - Objective function $f(\cdot)$: the objective to be minimized;
  - Constraint functions $f_i(\cdot)$: the constraints to be satisfied;
  - Feasible set $\mathcal{C}$: the set of all feasible points that satisfy all constraints,

$$\mathcal{C} \triangleq \{\boldsymbol{x} \in \mathcal{D} \mid f_i(\boldsymbol{x}) \leq 0, i = 1, ..., m\}.$$

- Convex Optimization Problem: an optimization problem with convex objective function and convex feasible set.

Unconstrained Convex Optimization

- Unconstrained Convex Optimization: a convex optimization problem without any constraint:

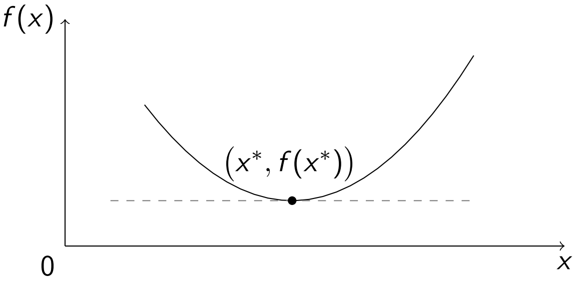
$$\text{minimize} \quad f(\boldsymbol{x})$$

Lemma (4.5)

Suppose $f(\cdot)$ is convex and differentiable. A feasible point $\boldsymbol{x}^* \in \mathcal{C}$ is a global minimizer of $f(\cdot)$ if and only if

$$\nabla f(\boldsymbol{x}^*)_i = \frac{\partial f(\boldsymbol{x}^*)}{\partial x_i} = 0, \quad \forall i = 1, ..., n.$$

Unconstrained Convex Optimization



Unconstrained Convex Optimization

- Computational Methods: find an algorithm that computes a sequence of feasible points $\boldsymbol{x}^{(0)}, \boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \ldots \boldsymbol{x}^{(k)}$, with

$$f(\boldsymbol{x}^{(k)}) \to f(\boldsymbol{x}^*) \quad \text{as} \quad k \to \infty$$

- Gradient-based Algorithms:

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} + \gamma^{(k)} \boldsymbol{d}^{(k)}$$

  - $\gamma^{(k)}$: a positive scalar (called step size) at iteration k;
  - $\boldsymbol{d}^{(k)}$: a gradient-based n-vector (called search direction) at iteration k;
  - Gradient Descent Method: $\boldsymbol{d}^{(k)} \triangleq -\nabla f(\boldsymbol{x}^{(k)})$
  - Newton's Method: $\boldsymbol{d}^{(k)} \triangleq -\left(\nabla^2 f(\boldsymbol{x}^{(k)})\right)^{-1} \nabla f(\boldsymbol{x}^{(k)})$

Constrained Convex Optimization

- Constrained Convex Optimization: a general convex optimization problem with convex constraints (i.e., $f_i(\cdot)$ function is convex for each i):

$$\begin{aligned} \text{minimize} \quad & f(\boldsymbol{x}) \\ \text{subject to} \quad & f_i(\boldsymbol{x}) \leq 0, \ i = 1, ..., m, \end{aligned}$$

Lemma (4.6)

Suppose $f(\cdot)$ is convex and differentiable. A feasible point $\boldsymbol{x}^* \in \mathcal{C}$ is a global minimizer of $f(\cdot)$ if and only if

$$\nabla f(\boldsymbol{x}^*)^T(\boldsymbol{x} - \boldsymbol{x}^*) \geq 0, \quad \forall \boldsymbol{x} \in \mathcal{C}.$$

Constrained Convex Optimization

- Geometrically, at a minimizer $\boldsymbol{x}^*$, the gradient $\nabla f(\boldsymbol{x}^*)$ makes an angle less than or equal to 90 degrees with all feasible variations $\boldsymbol{x} - \boldsymbol{x}^*$.
- Illustration of optimal $\boldsymbol{x}^*$:

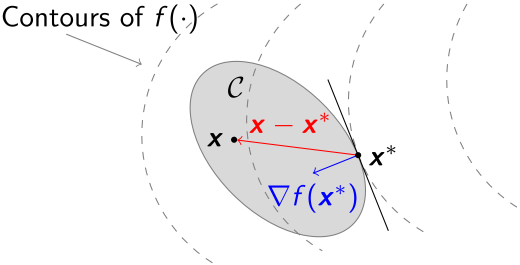


Figure: The gradient $\nabla f(\boldsymbol{x}^*)$ (blue arrow) makes an angle less than or equal to 90 degrees with all feasible variations $\boldsymbol{x} - \boldsymbol{x}^*$ (red arrow).

Constrained Convex Optimization

- Computational Methods: Gradient-based Algorithms:

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} + \gamma^{(k)} \boldsymbol{d}^{(k)},$$

  - Conditional Gradient Method:

$$\boldsymbol{d}^{(k)} \triangleq \overline{\boldsymbol{x}}^{(k)} - \boldsymbol{x}^{(k)},$$

    where $\overline{\boldsymbol{x}}^{(k)} \triangleq \arg\max_{\boldsymbol{x} \in \mathcal{C}} \nabla f(\boldsymbol{x}^{(k)})^T (\boldsymbol{x} - \boldsymbol{x}^{(k)})$ subject to $\nabla f(\boldsymbol{x}^{(k)})^T (\boldsymbol{x} - \boldsymbol{x}^{(k)}) < 0$.

  - Gradient Projection Method:

$$\boldsymbol{d}^{(k)} \triangleq \overline{\boldsymbol{x}}^{(k)} - \boldsymbol{x}^{(k)},$$

    where $\overline{\boldsymbol{x}}^{(k)}$ is given by $\overline{\boldsymbol{x}}^{(k)} \triangleq \left[\boldsymbol{x}^{(k)} - s^{(k)} \nabla f(\boldsymbol{x}^{(k)})\right]^+$. Here $[\cdot]^+$ denotes a projection on the feasible set $\mathcal{C}$, and $s^{(k)}$ is a positive scalar.

Duality Principle

- An important theoretical framework to solve convex optimization problems.
- Basic Idea: Convert the original optimization problem (called primal problem) into a dual problem.
    - The solution to the dual problem provides a lower bound to the solution of the primal problem.
    - Maximizing the objective of dual problem help us understanding the optimal objective of the primal problem.

Lagrange Function

- Recall the constrained optimization problem

$$\begin{aligned} \text{minimize} \quad & f(\boldsymbol{x}) \\ \text{subject to} \quad & f_i(\boldsymbol{x}) \leq 0, \;\; i = 1, ..., m, \end{aligned}$$

Definition (Lagrangian Function)

The Lagrangian function $L(\cdot) : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ is defined as

$$L(\boldsymbol{x}, \boldsymbol{\lambda}) \triangleq f(\boldsymbol{x}) + \sum_{i=1}^{m} \lambda_i f_i(\boldsymbol{x}).$$

- Intuitively, Lagrangian function is a weighted sum of the objective function $f(\boldsymbol{x})$ and the constraint functions $f_i(\boldsymbol{x})$.
- $\lambda_i \geq 0$: the weight (called Lagrange multiplier or dual variable) associated with each constraint $f_i(\boldsymbol{x}) \leq 0$.

Dual Function

Definition (Dual Function)

The Lagrange dual function $g : \mathbb{R}^m \to \mathbb{R}$ is defined as the minimum value of the Lagrangian function over $\boldsymbol{x}$:

$$g(\boldsymbol{\lambda}) \triangleq \inf_{\boldsymbol{x}} L(\boldsymbol{x}, \boldsymbol{\lambda}) = \inf_{\boldsymbol{x}} \left(f(\boldsymbol{x}) + \sum_{i=1}^{m} \lambda_i f_i(\boldsymbol{x}) \right).$$

- The dual function $g(\cdot)$ is always concave even if the primal problem is not convex.
- The dual function $g(\cdot)$ yields a lower bound of the optimal primal objective value $f(\boldsymbol{x}^*)$:

$$g(\boldsymbol{\lambda}) \leq f(\boldsymbol{x}^*), \quad \forall \boldsymbol{\lambda} \succeq \mathbf{0}$$

Lagrange Dual Problem

- The dual function $g(\boldsymbol{\lambda})$ yields lower bounds of the optimal primal objective value $f(\boldsymbol{x}^*)$.
  - How far the dual function $g(\boldsymbol{\lambda})$ is apart from the optimal $f(\boldsymbol{x}^*)$?
- Lagrange Dual Problem: find the optimal dual variables $\boldsymbol{\lambda}^*$ that maximizes the dual function $g(\boldsymbol{\lambda})$:

$$\begin{aligned} \text{maximize} \quad & g(\boldsymbol{\lambda}) \\ \text{subject to} \quad & \boldsymbol{\lambda} \succeq \mathbf{0}. \end{aligned}$$

  - Weak duality: $g(\boldsymbol{\lambda}^*) \leq f(\boldsymbol{x}^*)$. The difference $f(\boldsymbol{x}^*) - g(\boldsymbol{\lambda}^*)$ is called the optimal duality gap.
  - Strong duality: $g(\boldsymbol{\lambda}^*) = f(\boldsymbol{x}^*)$ if the optimality gap is zero.

Duality Gap

- Duality Gap: The gap between primal and dual objectives:

$$f(\boldsymbol{x}) - g(\boldsymbol{\lambda})$$

  - The duality gap reflects how suboptimal a given point $\boldsymbol{x}$ is, without knowing the exact value of $f(\boldsymbol{x}^*)$:

$$f(\boldsymbol{x}) - f(\boldsymbol{x}^*) \leq f(\boldsymbol{x}) - g(\boldsymbol{\lambda})$$

  - Any primal-dual feasible pair $\{\boldsymbol{x}, \boldsymbol{\lambda}\}$ localizes the optimal primal and dual objectives to an interval $[g(\boldsymbol{\lambda}), f(\boldsymbol{x})]$, that is,

$$g(\boldsymbol{\lambda}) \leq g(\boldsymbol{\lambda}^*) \leq f(\boldsymbol{x}^*) \leq f(\boldsymbol{x})$$

KKT Optimality Conditions

Lemma (Karush-Kuhn-Tucker (KKT) Conditions)

Assume that the primal problem is strictly convex and the strong duality holds. A primal-dual feasible pair $\{\boldsymbol{x}^*, \boldsymbol{\lambda}^*\}$ *is optimal for both primal and dual problems, if and only if*

$$\begin{cases} f_i(\boldsymbol{x}^*) \leq 0, \ \lambda_i^* \geq 0, \ \lambda_i^* \cdot f_i(\boldsymbol{x}^*) = 0, \quad i = 1, ..., m \\ \nabla f(\boldsymbol{x}^*) + \sum_{i=1}^{m} \lambda_i^* \nabla f_i(\boldsymbol{x}^*) = \mathbf{0}. \end{cases}$$

Shadow Price

- Shadow Price: A geometric interpretation of the Lagrange multipliers λ_i, $i = 1, ..., m$, in terms of economics.
  - Introduce perturbing parameters $\boldsymbol{u} \triangleq (u_i, i = 1, ..., m)$, and define a perturbed version of the original primal problem:

$$\begin{aligned} \text{minimize} \quad & f(\boldsymbol{x}) \\ \text{subject to} \quad & f_i(\boldsymbol{x}) \leq u_i, \; i = 1, ..., m \end{aligned}$$

  - Denote the optimal perturbed objective as $p^*(\boldsymbol{u}) = \inf_{\boldsymbol{x}} f(\boldsymbol{x})$:

$$\frac{\partial p^*(\mathbf{0})}{\partial u_i} = -\lambda_i^*$$

    - ★ $f(\boldsymbol{x})$: the total cost;
    - ★ x_i: the investment on resource i;
    - ★ u_i: the limit on resource i's investment;
  - When $\boldsymbol{u}$ is close to $\mathbf{0}$, the λ_i^* reflects how much more profit the firm could make, for a small increase in the availability of resource i.

Solving Dual Problem

- Subgradient: A vector $\boldsymbol{d}$ is called a subgradient of $f(\cdot)$ at a point $\boldsymbol{x}$, if

$$f(\boldsymbol{z}) \geq f(\boldsymbol{x}) + \boldsymbol{d}^T(\boldsymbol{z} - \boldsymbol{x}), \quad \forall \boldsymbol{z} \in \mathcal{D}(f).$$

- Subgradient method for solving the due problem
  - A subgradient $\boldsymbol{d}$ of the dual function $g(\boldsymbol{\lambda})$ at a point $\boldsymbol{\lambda}$ satisfies:

$$g(\boldsymbol{\mu}) \leq g(\boldsymbol{\lambda}) + \boldsymbol{d}^T(\boldsymbol{\mu} - \boldsymbol{\lambda}), \quad \forall \boldsymbol{\mu} \in \mathcal{D}(g).$$

  - Subgradient Method:

$$\boldsymbol{\lambda}^{(k+1)} = \left[\boldsymbol{\lambda}^{(k)} + \gamma^{(k)}\boldsymbol{d}^{(k)}\right]^+$$

Solving Dual Problem

Lemma

For every dual optimal solution $\boldsymbol{\lambda}^*$, *we have* $||\boldsymbol{\lambda}^{(k+1)} - \boldsymbol{\lambda}^*|| < ||\boldsymbol{\lambda}^{(k)} - \boldsymbol{\lambda}^*||$ *for all step-sizes* $\gamma^{(k)}$ *satisfying*

$$0 < \gamma^{(k)} < 2 \cdot \frac{g(\boldsymbol{\lambda}^*) - g(\boldsymbol{\lambda}^{(k)})}{||\boldsymbol{d}^{(k)}||^2}.$$

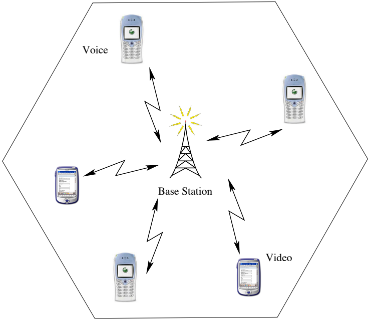
- The above range for $\gamma^{(k)}$ requires the dual optimal value $g(\boldsymbol{\lambda}^*)$, which is usually unknown.
- In practice, we can use the following approximate step-size formula

$$\gamma^{(k)} = \alpha^{(k)} \cdot \frac{g^{(k)} - g(\boldsymbol{\lambda}^{(k)})}{||\boldsymbol{d}^{(k)}||^2},$$

where $g^{(k)}$ is an approximation of $g(\boldsymbol{\lambda}^*)$, and $0 < \alpha^{(k)} < 2$.

Section 4.2: Resource Allocation for Wireless Video Streaming

Network Model



- A single cell CDMA network with mixed video and voice users.
- Voice users are background traffic: just need good enough channels.
- Video users can adapt to channel conditions, but with deadline constraints.

Network Optimization Problem

- Maximize the overall quality of video users, subject to the QoS constraints of the voice users.
- The general solution framework involve three phases
  1. Average resource allocation among video users
  2. Video source adaptions
  3. Multiuser deadline oriented scheduling
- We will focus on the formulation of Phase 1.

Average Resource Allocation

- A set $\mathcal{N} = \{1, \ldots, N\}$ video users.
- Each video user n has a utility function $u_n(x_n)$.
  - Increasing and strictly concave in the resource allocation x_n.
  - Corresponds to commonly used video quality measures such as the rate-PSNR function and rate-summarization distortion functions.
  - Assume $u_n(x_n)$ is a continuous and differentiable function.
- The network resource can be transmission power (uplink) or transmission time (downlink).

Network Utility Maximization (NUM) Problem

NUM Problem

$$\begin{aligned} \text{maximize} \quad & \sum_{n \in \mathcal{N}} u_n(x_n) \\ \text{subject to} \quad & \sum_{n \in \mathcal{N}} x_n \leq X_{\max}. \\ \text{variables} \quad & x_n \geq 0, \forall n \in \mathcal{N}. \end{aligned}$$

- We will solve this using the dual-based sub-gradient method.

Lagragian Relaxation

- Relax the constraint with a dual variable λ and obtain the Lagrangian

$$L(\boldsymbol{x}, \lambda) \triangleq \sum_n u_n(x_n) - \lambda\left(\sum_n x_n - X_{\max}\right).$$

- λ is the shadow price for the limited resource $X_{\max}$.

Dual-based Solution

- Solve the NUM problem at two levels (separation of time scales)
  - Lower level: each user n chooses x_n to maximize surplus:

$$\max_{x_n \geq 0} \quad u_n\left(x_n\right) - \lambda x_n, \tag{1}$$

    and the unique optimal solution is $x_n\left(\lambda\right)$. We further denote $g_n(\lambda)$ as the maximum objective value of Problem (1) for a given value of λ.
  - Higher level: The base station adjusts λ to solve the following problem

$$\min_{\lambda \geq 0} \quad L\left(\boldsymbol{x}(\lambda), \lambda\right) \triangleq \sum_n g_n\left(\lambda\right) + \lambda X_{\max},$$

    using the sub-gradient searching method,

$$\lambda^{(k+1)} = \max\left\{0, \lambda^{(k)} + \alpha^{(k)}\left(\sum_n x_n\left(\lambda^{(k)}\right) - X_{\max}\right)\right\}.$$

How to Model Wireless Resources

- 3G CDMA technology: users transmit using orthogonal codes
  - Uplink transmissions: from users to the base station, asynchronization transmissions leads to mutual interference among users
  - Downlink transmission: from base station to users, no mutual interference among users
- In both cases, need to model the resource constraint for the video users, given the voice users' QoS requirements

Wireless Uplink Streaming

- Consider M voice users and N video users, mutually interfering with each other
- A user's QoS is determined by the Signal-to-interference plus noise ratio (SINR)
- A voice user needs to achieve an SINR target of γ_{voice}:

$$\frac{W}{R_{voice}} \frac{G_{voice} P^r_{voice}}{n_0 W + (M-1) P^r_{voice} + P^{r,all}_{video}} \geq \gamma_{voice}.$$

  - W: total bandwidth
  - n_0: background noise density
  - R_{voice}: voice user's target data rate
  - G_{voice}: related to voice users' modulation and coding choices
  - P^r_{voice}: a voice user's received power at the base station
  - $P^{r,all}_{video}$: total video users' received power at the base station

Wireless Uplink Streaming

- To satisfy the target SINR for M voice users, we can derive the maximum total video users' received power at the base station

$$P_{video}^{r,\max} = \left(\frac{WG_{voice}}{R_{voice}\gamma_{voice}} - (M-1) \right) P_{voice}^{r} - n_0 W.$$

Wireless Uplink Streaming

- The NUM problem ⇒ video transmission power optimization problem during time $[0, T]$:

NUM Problem for Wireless Uplink Streaming - Version 1

$$\max_{\{p_n(t),\forall n\}} \sum_{n=1}^{N} u_n\left(\int_0^T r_n\left(\boldsymbol{p}\left(t\right)\right) dt\right)$$

$$\text{s.t.} \quad \sum_{n=1}^{N} h_n p_n\left(t\right) \leq P_{video}^{r,\max}, \forall t \in [0, T]$$

$$0 \leq p_n\left(t\right) \leq P_n^{\max}, \forall n, \forall t \in [0, T]$$

- $p_n(t)$: video user n's transmission power at time t.
- h_n: channel gain from the transmitter of user n to the base station.
- $P_n^{\max}$: maximum peak transmission power of user n.
- $r_n\left(\boldsymbol{p}(t)\right)$: data rate achieved by user n at time t, depending on all users' transmission power $\boldsymbol{p}(t)$.

Wireless Uplink Streaming

- Solving functions are challenging, hence needs further simplification.
- Assume video users transmit via time-division-multiplexing (TDM)
  - Video users take turns to transmit.
  - The constant data rate of video user n is

$$R_n^{TDM} = W \log_2 \left(1 + \frac{\min \{ h_n P_n^{\max}, P_{video}^{r,\max} \}}{n_0 W + M P_{voice}^{r}} \right).$$

- The NUM problem $\Rightarrow$ the transmission time optimization problem

NUM Problem for Wireless Uplink Streaming -Version 2

$$\max_{\{t_n \geq 0, \forall n\}} \sum_{n=1}^{N} u_n \left(R_n^{TDM} t_n \right), \text{ s.t. } \sum_{n=1}^{N} t_n \leq T.$$

  - t_n: transmission time of video user n.

Wireless Downlink Streaming

- Orthogonal transmission without mutual interferences
- Video users can transmit simultaneously
- A video user n transmits with power p_n and achieves a data rate

$$r_n(p_n) = W \log_2 \left(1 + \frac{h_n p_n}{n_0 W}\right).$$

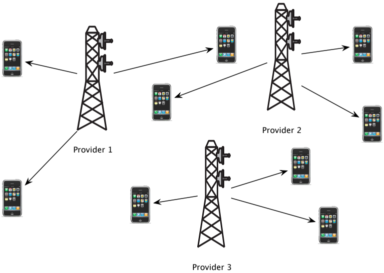
- The NUM problem $\Rightarrow$ the transmission power optimization problem

NUM Problem for Wireless Downlink Streaming

$$\max_{\{p_n \geq 0, \forall n\}} \sum_{n=1}^{N} u_n \left(T \cdot r_n(p_n)\right), \text{ s.t. } \sum_{n=1}^{N} p_n \leq P_{\max}^{video}.$$

Section 4.3: Wireless Service Provider Pricing

Network Model



- A set $\mathcal{J} = \{1, \ldots, J\}$ of service providers
  - Provider j has a supply Q_j of resource (e.g., channel, time, power)
  - Providers operate on orthogonal spectrum bands
- A set $\mathcal{I} = \{1, \ldots, I\}$ of users
  - User i can obtain resources from multiple providers: $\boldsymbol{q}_i = (q_{ij}, \forall j \in \mathcal{J})$
  - User i's utility function is $u_i\left(\sum_{j=1}^{J} q_{ij} c_{ij}\right)$: increasing and strictly concave

An Example: TDMA

- Each provider j has a total spectrum band of W_j.
- q_{ij}: the fraction of time that user i transmits on provider j's band
  - Constraints: $\sum_i q_{ij} \leq 1$, for all $j \in \mathcal{J}$.
- c_{ij}: the data rate achieved by user i on provider j's band

$$c_{ij} = W_j \log(1 + \frac{P_i |h_{ij}|^2}{\sigma_{ij}^2 W_j})$$

  - P_i: user i's peak transmission power.
  - h_{ij}: the channel gain between user i and network j.
  - σ_{ij}^2: the Gaussian noise variance for the channel.
- $u_i \left(\sum_{j=1}^{J} q_{ij} c_{ij} \right)$: user i' utility of the total achieved data rate

Social Welfare Optimization

- $x_i(\boldsymbol{q}_i)$: effective resource obtained by use i

$$x_i(\boldsymbol{q}_i) = \sum_{j=1}^{J} q_{ij} c_{ij}$$

SWO: Social Welfare Optimization Problem

$$\begin{aligned} \text{maximize} \quad & \sum_{i \in \mathcal{I}} u_i(x_i) \\ \text{subject to} \quad & \sum_{j \in \mathcal{J}} q_{ij} c_{ij} = x_i, \ \forall i \in \mathcal{I}, \\ & \sum_{i \in \mathcal{I}} q_{ij} = Q_j, \ \forall j \in \mathcal{J}, \\ \text{variables} \quad & q_{ij}, x_i \geq 0, \ \forall i \in \mathcal{I}, j \in \mathcal{J}. \end{aligned}$$

Social Welfare Optimization

- We can just consider variables $\boldsymbol{q}$ in SWO, since $\boldsymbol{q}$ determines $\boldsymbol{x}$.
  - Why not?
- SWO is a strictly concave maximization problem in $\boldsymbol{x}$.
  - A unique optimal solution $\boldsymbol{x}^*$
- SWO is not strictly concave maximization problem in $\boldsymbol{q}$
  - The optimal solution $\boldsymbol{q}^*$ may not be unique
  - But we can show that $\boldsymbol{q}^*$ is unique (with probability 1) if c_{ij}'s are continuous random variables.

Solving SWO Problem

- We can use the dual-based sub gradient algorithm
- Next we introduce the primal-dual based algorithm

Primal-Dual Algorithm

- Key idea: updating primal and dual variables simultaneously using small step sizes
- No longer requires separate of time scales.
- Suitable when it is not easy to solve the optimal primary variables under fixed dual prices.

Some Definitions

- $f_{ij}(t)$ (or simply f_{ij}): the marginal utility of user i with respect to q_{ij} when his demand vector is $\boldsymbol{q_i}(t)$:

$$f_{ij} = \frac{\partial u_i(\boldsymbol{q_i})}{\partial q_{ij}} = c_{ij} \frac{\partial u_i(x)}{\partial x}\Big|_{x=x_i=\sum_{j=1}^{J} q_{ij} c_{ij}}$$

- $(x)^+ = \max(0, x)$ and

$$(x)_y^+ = \begin{cases} x & y > 0 \\ (x)^+ & y \leq 0. \end{cases}$$

Primal-Dual Algorithm

Continuous Time Primal-Dual Algorithm

$$\dot{q}_{ij} = k_{ij}^{q} \left(f_{ij} - p_j \right)_{q_{ij}}^{+}, \ \forall i \in \mathcal{I}, \forall j \in \mathcal{J},$$

$$\dot{p}_j = k_j^{p} \left(\sum_{i=1}^{I} q_{ij} - Q_j \right)_{p_j}^{+}, \ \forall j \in \mathcal{J}.$$

- k_{ij}^{p}'s and k_j^{p}'s: constants representing update rates.
- A user will increase resource request when marginal utility is larger than price.
- A provider will increase the price is the total demand is larger than the supply.
- When a variables (q_{ij} or p_j) is zero, it will not become negative even when the direction of the update is negative.

Convergence of Primal-Dual Algorithm

- First, construct a La Salle function $V(\boldsymbol{q}(t), \boldsymbol{p}(t))$:

$$
\begin{aligned}
V(t) &= V(\boldsymbol{q}(t), \boldsymbol{p}(t)) \\
&= \sum_{i,j} \frac{1}{k_{ij}^q} \int_0^{q_{ij}(t)} (\beta - q_{ij}^*) d\beta + \sum_j \frac{1}{k_j^p} \int_0^{p_j(t)} (\beta - p_j^*) d\beta.
\end{aligned}
$$

- Second, show $V(\boldsymbol{q}(t), \boldsymbol{p}(t))$ is non-increasing for any solution trajectory $(\boldsymbol{q}(t), \boldsymbol{p}(t))$ that following the primal-dual algorithm, i.e.,

$$
\dot{V}(t) = \sum_{i,j} \frac{\partial V}{\partial q_{ij}} \dot{q}_{ij} + \sum_j \frac{\partial V}{\partial p_j} \dot{p}_j,
$$

  is always nonpositive.

- Since $V(t)$ is lower bounded, the algorithm converges.

Numerical Example

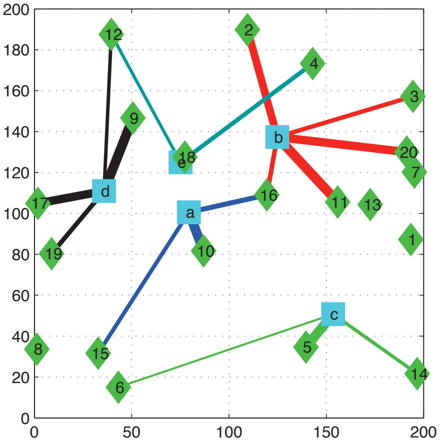


Figure: Example of equilibrium user-provider association. The users are labeled by numbers (1-20), and the providers are labeled by letters (*a*-*e*).

Numerical Example

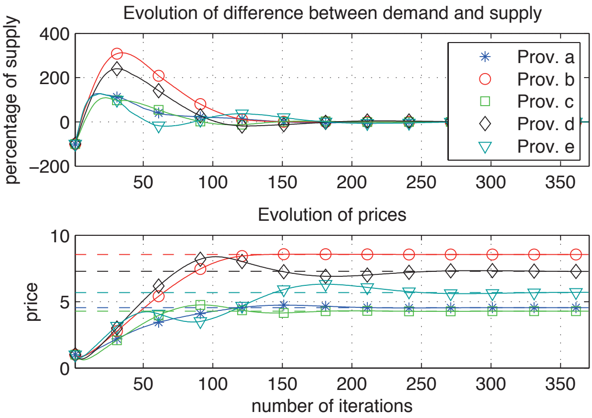


Figure: Evolution of the primal-dual algorithm

Section 4.4: Chapter Summary

Key Concepts

- Theory
  - Convex set
  - Convex function
  - Convex optimization
  - Duality
  - Dual-based sub gradient algorithm
  - Primal-dual algorithm
- Application
  - Resource Allocation for Wireless Video Streaming
  - Wireless Service Provider Pricing

References and Extended Reading

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