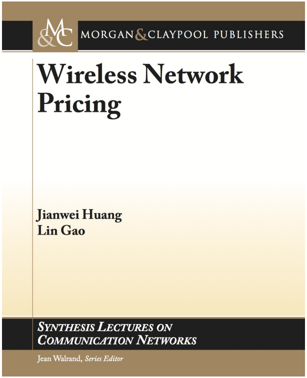
Wireless Network Pricing
Chapter 6: Oligopoly Pricing

Jianwei Huang & Lin Gao

Network Communications and Economics Lab (NCEL)
Information Engineering Department
The Chinese University of Hong Kong

The Book



- E-Book freely downloadable from NCEL website: `http://ncel.ie.cuhk.edu.hk/content/wireless-network-pricing`
- Physical book available for purchase from Morgan & Claypool (`http://goo.gl/JFGlai`) and Amazon (`http://goo.gl/JQKaEq`)

Chapter 6: Oligopoly Pricing

Focus of This Chapter

- Key Focus: This chapter focuses on the user interactions in an oligopoly market, where multiple self-interested individuals make decisions independently, and the payoff of each individual depends not only on his own decision, but also on the decisions of others.
- Theoretic Approach: Game Theory
  - Strategic Form Game
  - Extensive Form Game

Game Theory

- Follow the discussions in
  - "*A course in game theory*" by M. Osborne and A. Rubinstein, 1994;
  - "*A Primer in Game Theory*" by R. Gibbons, 1992;
  - "*Game theory with applications to economics*" by J. Friedman, 1986;
  - "*Game theory and applications*" by L. Petrosjan and V. Mazalov, 2002.

Definition (Game Theory)

Game theory is a study of strategic decision making. Specifically, it is the study of *mathematical models of conflict and cooperation* between intelligent rational individuals.

Section 6.1
Theory: Game Theory

What is a game?

- A game is a formal representation of a situation in which a number of individuals interact with strategic interdependence.
  - *Each individual's payoff depends not only on his own choice, but also on the choices of other individuals;*
  - *Each individual is rational (self-interested), whose goal is to choose the actions that produce his most preferred outcome.*

- Key components of game
  - Players: Who are involved in the game?
  - Rules: What actions can players choose? How and when do they make decisions? What information do players know about each other when making decisions?
  - Outcomes: What is the outcome of the game for each possible action combinations chosen by players?
  - Payoffs: What are the players' preferences (i.e., utilities) over the possible outcomes?

Strategic Form Game

- In strategic form games (also called normal form games), all players make decisions simultaneously without knowing each other's choices.

Definition (Strategic Form Game)

A strategic form game is a triplet $\langle \mathcal{I}, (\mathcal{S}_i)_{i\in\mathcal{I}}, (u_i)_{i\in\mathcal{I}} \rangle$ where

- $\mathcal{I} = \{1, 2, ..., I\}$ is a finite set of players;
- $\mathcal{S}_i$ is a set of available actions (pure strategies) for player $i \in \mathcal{I}$;
  - $\mathbb{S} \triangleq \Pi_i \mathcal{S}_i$ denotes the set of all action profiles.
- $u_i : \mathbb{S} \to \mathbb{R}$ is the payoff (utility) function of player i, which maps every possible action profile in $\mathbb{S}$ to a real number.

Strategic Form Game

- Strictly Dominated Strategy
  - A strictly dominated strategy refers to a strategy that is always worse than all other strategies of the same player regardless of the choices of other players'.
  - A strictly dominated strategy can be safely removed from the player's strategy set without changing the game outcome.

Definition (Strictly Dominated Strategy)

A strategy $s_i \in \mathcal{S}_i$ is strictly dominated for player i, if there exists some $s_i' \in \mathcal{S}_i$ such that

$$u_i(s_i, \boldsymbol{s}_{-i}) < u_i(s_i', \boldsymbol{s}_{-i}), \quad \forall \boldsymbol{s}_{-i} \in \mathbb{S}_{-i}.$$

Strategic Form Game

- Example: Prisoner's Dilemma Game
  - Two players are arrested for a crime and placed in separate rooms. The authorities try to extract a confession from them;
  - Strategy of each player: SILENT, CONFESS;
  - Payoff of players:

	SILENT	CONFESS
SILENT	$(-2, -2)$	$(-5, -1)$
CONFESS	$(-1, -5)$	$(-4, -4)$

    - ⋆ Each row denotes one action of player 1, each column denotes one action of player 2.

  - *"SILENT" is a strictly dominated strategy for both players.*

Strategic Form Game

- Best Response Correspondence
  - A best response is the strategy which produces the most preferred outcome for a player, taking all other players' strategies as given.

Definition (Best Response Correspondence)

For each player i, the best response correspondence $B_i(\boldsymbol{s}_{-i}) : \mathbb{S}_{-i} \to \mathcal{S}_i$ is a mapping from the set $\mathbb{S}_{-i}$ into $\mathcal{S}_i$ such that

$$B_i(\boldsymbol{s}_{-i}) = \{s_i \in \mathcal{S}_i \mid u_i(s_i, \boldsymbol{s}_{-i}) \geq u_i(s_i', \boldsymbol{s}_{-i}), \forall s_i' \in \mathcal{S}_i\}.$$

- $\boldsymbol{s}_{-i} = (s_j, \forall j \neq i)$ is the vector of actions for all players except i;
- $\mathbb{S}_{-i} \triangleq \Pi_{j \neq i} \mathcal{S}_j$ is the set of action profiles for all players except i.

Strategic Form Game

- Example: Stag Hunt Game
  - Two hunters (players) decide to hunt together in a forest, and each of them chooses one animal to hunt;
  - Strategy of each player: STAG, HARE;
  - Payoff of players:

	STAG	HARE
STAG	$(10, 10)$	$(0, 2)$
HARE	$(2, 0)$	$(2, 2)$

    - ★ Each row denotes one action of player 1, each column denotes one action of player 2.

  - *No strictly dominated strategy in this game;*
  - *If one player chooses the strategy "STAG", the best strategy of the other player is also "STAG";*
  - *If one player chooses the strategy "HARE", the best strategy of the other player is also "HARE".*

Strategic Form Game

- Nash Equilibrium
  - A Nash equilibrium is such a strategy profile under which no player has the incentive to change his strategy unilaterally.

Definition (Pure Strategy Nash Equilibrium)

A pure strategy Nash Equilibrium of a strategic form game $\langle \mathcal{I}, (\mathcal{S}_i)_{i\in\mathcal{I}}, (u_i)_{i\in\mathcal{I}}\rangle$ is a strategy profile $\boldsymbol{s}^* \in \mathbb{S}$ such that for each player $i \in \mathcal{I}$, the following condition holds

$$u_i(s_i^*, \boldsymbol{s}_{-i}^*) \geq u_i(s_i', \boldsymbol{s}_{-i}^*), \quad \forall s_i' \in \mathcal{S}_i.$$

Strategic Form Game

- A strategy profile $\boldsymbol{s}^* \in \mathbb{S}$ is a pure strategy Nash Equilibrium if and only if

$$s_i^* \in B_i(\boldsymbol{s}_{-i}^*), \quad \forall i \in \mathcal{I}.$$

- In the example of Prisoner's Dilemma Game, there is one pure strategy Nash Equilibrium: (CONFESS, CONFESS);
- In the example of Stag Hunt Game, there are two pure strategy Nash Equilibriums: (STAG, STAG) and (HARE, HARE).

Strategic Form Game

- A game may have no pure strategy Nash Equilibrium.
- Example: Matching Pennies Game
  - Two players turn their pennies to "HEADS" or "TAILS" secretly and simultaneously;
  - Strategy of each player: HEADS, TAILS;
  - Payoff of players:

	HEADS	TAILS
HEADS	$(1,-1)$	$(-1,1)$
TAILS	$(-1,1)$	$(1,-1)$

    - ★ Each row denotes one action of player 1, each column denotes one action of player 2.

  - *No pure strategy Nash equilibrium in this game;*
  - *A Natural Question: What kind of outcome will emerge as an "equilibrium"?* $\rightarrow$ *Mixed Strategy Nash Equilibrium*

Strategic Form Game

- Mixed Strategy
  - A mixed strategy is a probability distribution function (or probability mass function) over all pure strategies of a player.
  - For example, in the Matching Pennies Game, a mixed strategy of player 1 is $\sigma_1 = (0.4, 0.6)$, which means that player 1 picks "HEADS" with probability 0.4 and "TAILS" with probability 0.6.
  - Expected Payoff under Mixed Strategy

$$u_i(\boldsymbol{\sigma}) = \sum_{\boldsymbol{s} \in \mathbf{S}} \left(\Pi_{j=1}^{I} \sigma_j(s_j) \right) \cdot u_i(\boldsymbol{s}),$$

    - ★ $\boldsymbol{\sigma} = (\sigma_j, \forall j \in \mathcal{I})$ is a mixed strategy profile;
    - ★ $\boldsymbol{s} = (s_j, \forall j \in \mathcal{I})$ is a pure strategy profile;
    - ★ $\sigma_j(s_j)$ is the probability of player j choosing pure strategy s_j.

Strategic Form Game

- Mixed Strategy Nash Equilibrium
  - A mixed strategy Nash equilibrium is such a mixed strategy profile under which no player has the incentive to change his mixed strategy unilaterally.

Definition (Mixed Strategy Nash Equilibrium)

A mixed strategy profile $\boldsymbol{\sigma}^*$ is a mixed strategy Nash Equilibrium if for every player $i \in \mathcal{I}$,

$$u_i(\sigma_i^*, \boldsymbol{\sigma}_{-i}^*) \geq u_i(\sigma_i', \boldsymbol{\sigma}_{-i}^*), \quad \forall \sigma_i' \in \Sigma_i.$$

- In the example of Matching Pennies Game, there is one mixed strategy Nash Equilibrium: $\boldsymbol{\sigma}^* = (\sigma_1^*, \sigma_2^*)$ with $\sigma_i^* = (0.5, 0.5)$, $i = 1, 2$.

Strategic Form Game

- "Support" of Mixed Strategy
  - The "support" of a mixed strategy σ_i is the set of pure strategies which are assigned positive probabilities. That is, $\mathrm{supp}(\sigma_i) \triangleq \{s_i \in \mathcal{S}_i \mid \sigma_i(s_i) > 0\}$.

Theorem

A mixed strategy profile $\boldsymbol{\sigma}^*$ is a mixed strategy Nash Equilibrium if and only if for every player $i \in \mathcal{I}$, the following two conditions hold:

- *Every chosen action is equally good, that is, the expected payoff given $\boldsymbol{\sigma}^*_{-i}$ of every $s_i \in \mathrm{supp}(\sigma_i)$ is the same;*
- *Every non-chosen action is no better, that is, the expected payoff given $\boldsymbol{\sigma}^*_{-i}$ of every $s_i \notin \mathrm{supp}(\sigma_i)$ must be no larger than the expected payoff of $s_i \in \mathrm{supp}(\sigma_i)$.*

Strategic Form Game

- Existence of Nash Equilibrium
  - When or whether a strategic form game possesses a pure or mixed strategy Nash equilibrium?

Theorem (Existence (Nash 1950))

Any finite strategic game, i.e., a game that has a finite number of players and each player has a finite number of action choices, has at least one mixed strategy Nash Equilibrium.

Strategic Form Game

Theorem (Existence (Debreu-Fan-Glicksburg 1952))

The strategic form game $\langle \mathcal{I}, (\mathcal{S}_i)_{i\in\mathcal{I}}, (u_i)_{i\in\mathcal{I}} \rangle$ has a pure strategy Nash equilibrium, if for each player $i \in \mathcal{I}$ the following condition hold:

- *$\mathcal{S}_i$ is a non-empty, convex, and compact subset of a finite-dimensional Euclidean space.*
- *$u_i(\boldsymbol{s})$ is continuous in $\boldsymbol{s}$ and quasi-concave in s_i.*

- Compact: closed and bounded.
- Quasi-concave: a function $f(\cdot)$ is quasi-concave if $-f(\cdot)$ is quasi-convex
  - http://en.wikipedia.org/wiki/Quasiconvex_function

Extensive Form Game

- In extensive form games (also called normal form games), players make decisions sequentially.
- Our focus is on the multi-stage game with observed actions where:
  - All previous actions (called history) are observed, i.e., each player is perfectly informed of all previous events;
  - Some players may move simultaneously within the same stage.

Extensive Form Game

Definition (Extensive Form Game)

An extensive form game consists of four main elements:

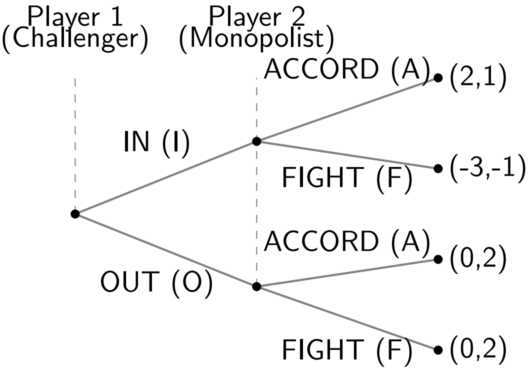
- A set of players $\mathcal{I} = \{1, 2, ..., I\}$;
- The history $\boldsymbol{h}^{k+1} = (\boldsymbol{s}^0, ..., \boldsymbol{s}^k)$ after each stage k, where $\boldsymbol{s}^t = (s_i^t, \forall i \in \mathcal{I})$ is the action profile at stage t;
- Each pure strategy for player i is defined as a contingency plan for every possible history after each stage;
- Payoffs are defined on the outcome after the last stage.

Extensive Form Game

- Important Notations
  - $\boldsymbol{h}^{k+1} = (\boldsymbol{s}^0, ..., \boldsymbol{s}^k)$: the history after stage k (i.e., at stage $k+1$);
  - $\mathcal{H}^{k+1} = \{\boldsymbol{h}^{k+1}\}$: the set of all possible histories after stage k;
  - $\mathcal{S}_i(\boldsymbol{h}^{k+1})$: the set of actions available to player i under a particular history $\boldsymbol{h}^k$ at stage $k+1$;
  - $\mathcal{S}_i(\mathcal{H}^{k+1}) = \bigcup_{\boldsymbol{h}^{k+1} \in \mathcal{H}^{k+1}} \mathcal{S}_i(\boldsymbol{h}^{k+1})$: the set of actions available to player i under all possible histories at stage $k+1$;
  - $a_i^k : \mathcal{H}^k \to \mathcal{S}_i(\mathcal{H}^k)$: a mapping from every possible history in $\mathcal{H}^k$ (after stage $k-1$) to an available action of player i in $\mathcal{S}_i(\mathcal{H}^k)$;
  - $s_i = \{a_i^k\}_{k=0}^{\infty}$: the pure strategy of player i.

Extensive Form Game

- Example: Market Entry Game
  - Two players: Player 1 (Challenger) and Player 2 (Monopolist);
    - ★ Player 1 chooses to enter the market (I) or stay out (O) at stage I;
    - ★ Player 2, after observing the action of Player 1, chooses to accommodate (A) or fight (F) at stage II;
  - Payoffs are illustrated on the leaf nodes after stage II.



Extensive Form Game

- Example: Market Entry Game
  - The strategy of Player 1: I, O;
  - The strategy of Player 2: AA, AF, FA, FF;
    - ★ AA: Player 2 will select "A" under both histories $\boldsymbol{h}^1 = \{\text{I}\}$ and $\{\text{O}\}$;
    - ★ AF: Player 2 will select "A" (or "F") under history $\boldsymbol{h}^1 = \{\text{I}\}$ (or $\{\text{O}\}$);
    - ★ FA: Player 2 will select "F" (or "A") under history $\boldsymbol{h}^1 = \{\text{I}\}$ (or $\{\text{O}\}$);
    - ★ FF: Player 2 will select "F" under both histories $\boldsymbol{h}^1 = \{\text{I}\}$ and $\{\text{O}\}$;
  - We can represent the extensive form game in the corresponding strategic form:

	AA	AF	FA	FF
I	$(2, 1)$	$(2, 1)$	$(-3, -1)$	$(-3, -1)$
O	$(0, 2)$	$(0, 2)$	$(0, 2)$	$(0, 2)$

  - ★ *Four Nash Equilibriums: (I, AA), (I, AF), (O, FA), and (O, FF)*
  - ★ *(O,FA) and (O,FF) are irreasonable, as they rely on the empty threat that Player 2 will choose "FIGHT" when player 1 chooses "IN".*

Extensive Form Game

- *How to characterize the reasonable Nash equilibrium in an extensive form game?* → Subgame Perfect Equilibrium

Definition (Subgame)

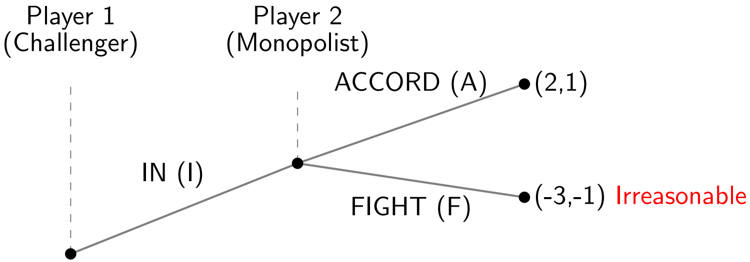
A subgame from history $\boldsymbol{h}^k$ is a game on which:

- Histories: $\boldsymbol{h}^{K+1} = (\boldsymbol{h}^k, \boldsymbol{s}^k, ..., \boldsymbol{s}^K)$.
- Strategies: $s_{i|\boldsymbol{h}^k}$ is the restriction of s_i to histories in $G(\boldsymbol{h}^k)$.
- Payoffs: $u_i(s_i, \boldsymbol{s}_{-i}|\boldsymbol{h}^k)$ is the payoff of player i after histories in $G(\boldsymbol{h}^k)$.

- A strategy profile $\boldsymbol{s}^*$ is a subgame perfect equilibrium if for every history $\boldsymbol{h}^k$, $s^*_{i|\boldsymbol{h}^k}$ is an Nash equilibrium of the subgame $G(\boldsymbol{h}^k)$.

Extensive Form Game

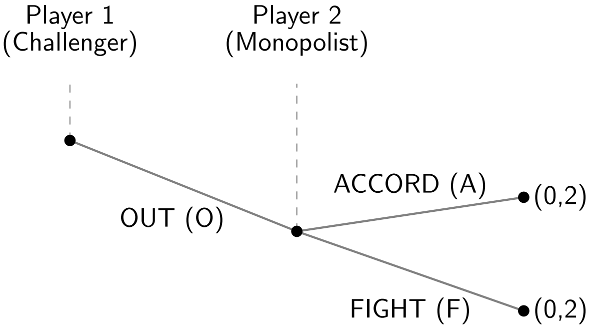
- Example: Market Entry Game
  - Subgame from History $\boldsymbol{h}^1 = \{\mathrm{I}\}$:



  - *In this subgame, Player 2 will always choose "ACCORD" (as 1 is better than -1), and hence we can eliminate "FIGHT".*

Extensive Form Game

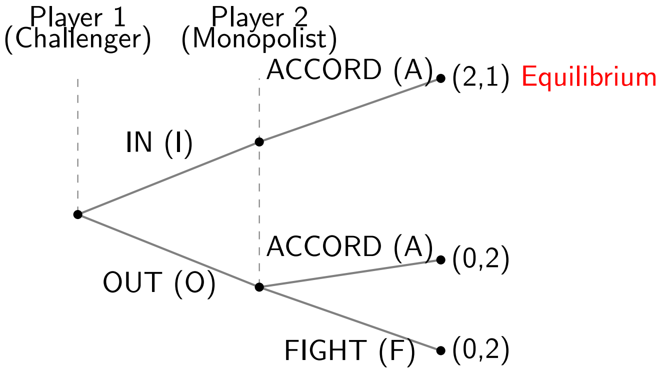
- Example: Market Entry Game
  - Subgame from History $\boldsymbol{h}^1 = \{O\}$:



  - *In this subgame, Player 2 is indifferent from choosing "ACCORD" or "FIGHT", hence we can not eliminate any action.*

Extensive Form Game

- Example: Market Entry Game
  - Player 1's action at stage I:
    - *IN: his payoff is 2 (as Player 2 will choose "ACCORD");*
    - *OUT: his payoff is 0 (no matter what Player 2 will choose).*
  - Equilibrium: Player 1 chooses "IN", Player 2 chooses "ACCORD".



Section 6.2
Theory: Oligopoly

Oligopoly

- In this part, we consider three classical strategic form games to formulate the interactions among multiple competitive entities (Oligopoly):
  - The Cournot Model
  - The Bertrand Model
  - The Hotelling Model
- Our purpose in this part is to illustrate
  - (a) Game Formulation: the translation of an informal problem statement into a strategic form representation of a game;
  - (b) Equilibrium Analysis: the analysis of Nash equilibrium when a player can choose his strategy from a continuous set.

The Cournot Model

- The Cournot model describes interactions among firms that *compete on the amount of output they will produce*, which they decide independently of each other simultaneously.
- Key features
  - At least two firms producing homogeneous products;
  - Firms do not cooperate, i.e., there is no collusion;
  - Firms compete by setting production quantities simultaneously;
  - The total output quantity affects the market price;
  - The firms are economically rational and act strategically, seeking to maximize profits given their competitors' decisions.

The Cournot Model

- Example: The Cournot Game
  - Two firms decide their respective output quantities simultaneously;
  - The market price is a decreasing function of the total quantity.
- Game Formulation
  - The set of players is $\mathcal{I} = \{1, 2\}$,
  - The strategy set available to each player $i \in \mathcal{I}$ is the set of all nonnegative real numbers, i.e., $q_i \in [0, \infty)$,
  - The payoff received by each player i is a function of both players' strategies, defined by

$$\Pi_i(q_i, q_{-i}) = q_i \cdot P(q_i + q_{-i}) - c_i \cdot q_i$$

    - ★ The first term denotes the player i's revenue from selling q_i units of products at a market-clearing price $P(q_i + q_{-i})$;
    - ★ The second term denotes the player i's production cost.

The Cournot Model

- Consider a linear cost: $P(q_i + q_{-i}) = a - (q_i + q_{-i})$
- Equilibrium Analysis
  - Given player 2's strategy q_2, the best response of player 1 is:

$$q_1^* = B_1(q_2) = \frac{a - q_2 - c_1}{2},$$

  - Given player 1's strategy q_1, the best response of player 2 is:

$$q_2^* = B_2(q_1) = \frac{a - q_1 - c_2}{2},$$

  - A strategy profile (q_1^*, q_2^*) is an Nash equilibrium if every player's strategy is the best response to others' strategies:

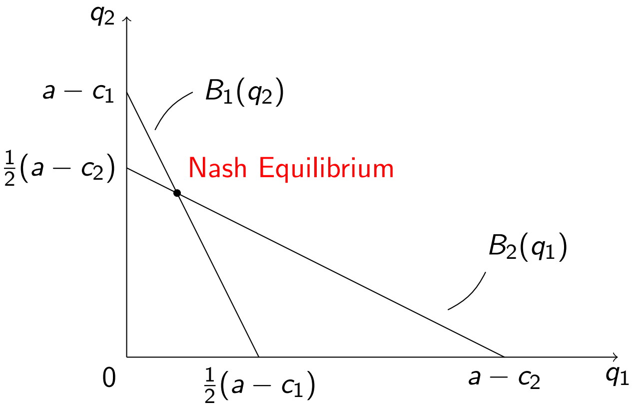
$$q_1^* = B_1(q_2^*), \quad \text{and} \quad q_2^* = B_2(q_1^*)$$

  - Nash Equilibrium:

$$q_1^* = \frac{a + c_1 + c_2}{3} - c_1, \quad q_2^* = \frac{a + c_1 + c_2}{3} - c_2$$

The Cournot Model

- Illustration of Equilibrium
  - Geometrically, the Nash equilibrium is the intersection of both players' best response curves.



The Bertrand Model

- The Bertrand model describes interactions among firms (sellers) who *set prices independently and simultaneously*, under which the customers (buyers) choose quantities accordingly.
- Key features
  - At least two firms producing homogeneous products;
  - Firms do not cooperate, i.e., there is no collusion;
  - Firms compete by setting prices simultaneously;
  - Consumers buy products from a firm with a lower cost (price).
    - If firms charge the same price, consumers randomly select among them.
  - The firms are economically rational and act strategically, seeking to maximize profits given their competitors' decisions.

The Bertrand Model

- Example: The Bertrand Game
  - Two firms decide their respective prices simultaneously;
  - The consumers buy products from a firm with a lower price.

- Game Formulation
  - The set of players is $\mathcal{I} = \{1, 2\}$,
  - The strategy set available to each player $i \in \mathcal{I}$ is the set of all nonnegative real numbers, i.e., $p_i \in [0, \infty)$,
  - The payoff received by each player i is a function of both players' strategies, defined by

$$\Pi_i(p_i, p_{-i}) = (p_i - c_i) \cdot D_i(p_1, p_2)$$

    - ★ c_i is the unit producing cost;
    - ★ $D_i(p_1, p_2)$ is the consumers' demand to player i: (i) $D_i(p_1, p_2) = 0$ if $p_i > p_{-i}$; (ii) $D_i(p_1, p_2) = D(p_i)$ if $p_i < p_{-i}$; (iii) $D_i(p_1, p_2) = D(p_i)/2$ if $p_i = p_{-i}$.

The Bertrand Model

- Equilibrium Analysis
  - Given player 2's strategy p_2, the best response of player 1 is to select a price p_1 slightly lower than p_2 under the constraint that $p_1 \geq c_1$:

$$p_1^* = \max\{c_1,\ p_2 - \epsilon\}$$

  - Given player 1's strategy p_1, the best response of player 2 is to select a price p_2 slightly lower than p_1 under the constraint that $p_2 \geq c_2$:

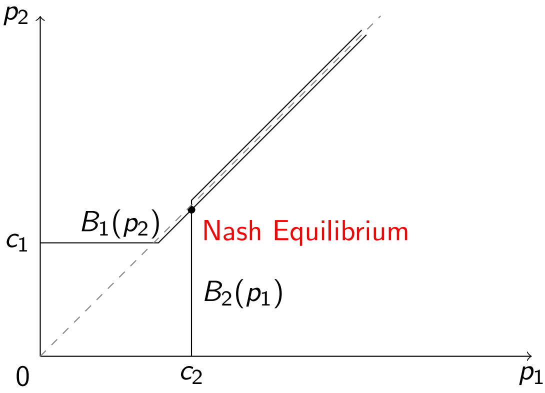
$$p_2^* = \max\{c_2,\ p_1 - \epsilon\}$$

  - Both players will gradually decrease their prices, until one player gets to his producing cost. Therefore, the Nash equilibrium is

$$\begin{cases} p_1^* = [c_2]^-, \quad p_2^* \in [c_2, \infty) & \text{if } c_1 < c_2 \\ p_1^* \in [c_1, \infty), \quad p_2^* = [c_1]^- & \text{if } c_1 > c_2 \\ p_1^* = \ p_2^* = c & \text{if } c_1 = c_2 = c \end{cases}$$

The Bertrand Model

- Illustration of Equilibrium
  - Geometrically, the Nash equilibrium is the intersection of both players' best response curves.



The Hotelling Model

- The Hotelling model studies *the effect of locations on the price competition* among two or more firms.
- Key features
  - Two firms at different locations sell the homogeneous good;
  - The customers are uniformly distributed between two firms.
  - Customers incur a transportation cost as well as a purchasing cost.
  - The firms are economically rational and act strategically, seeking to maximize profits given their competitors' decisions.

The Hotelling Model

- Example: The Hotelling Game
  - Two firms at different locations decide their respective prices simultaneously;
  - The consumers buy products from a firm with a lower total cost, including both the transportation cost and the purchasing cost.

- Game Formulation
  - The set of players is $\mathcal{I} = \{1, 2\}$, each locating at one end of the interval [0, 1];
  - The strategy set available to each player $i \in \mathcal{I}$ is the set of all nonnegative real numbers, i.e., $p_i \in [0, \infty)$;
  - The payoff received by each player i is a function of both players' strategies, defined by

$$\Pi_i(p_i, p_{-i}) = (p_i - c_i) \cdot D_i(p_1, p_2)$$

    - ★ c_i is the unit producing cost;
    - ★ $D_i(p_1, p_2)$ is the ratio of consumers coming to player i.

The Hotelling Model

- Consumer Demand: $D_i(p_1, p_2)$
  - Under price profile (p_1, p_2), the total cost of a consumer at location $x \in [0, 1]$ buying products from player 1 or 2 is

$$C_1(x) = p_1 + w \cdot x, \quad \text{and} \quad C_2(x) = p_1 + w \cdot (1 - x)$$

  - Under (p_1, p_2), two players receive the following consumer demand:

$$D_1(p_1, p_2) = \frac{p_2 - p_1 + w}{2w}, \quad D_2(p_1, p_2) = \frac{p_1 - p_2 + w}{2w}$$

The Hotelling Model

- Equilibrium Analysis
  - Given player 2's strategy p_2, the best response of player 1 is

$$p_1^* = B_1(p_2) = \frac{p_2 + w + c_1}{2}$$

  - Given player 1's strategy p_1, the best response of player 2 is

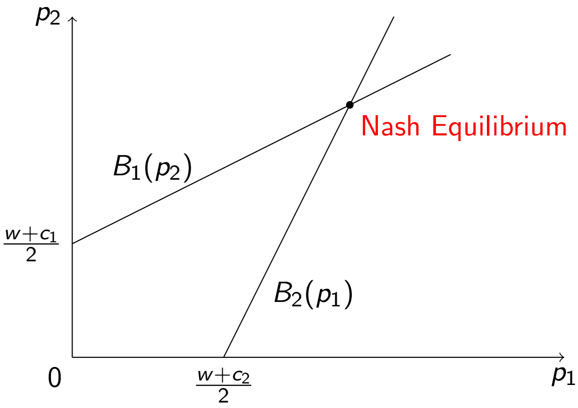
$$p_2^* = B_2(p_1) = \frac{p_1 + w + c_2}{2}$$

  - Nash Equilibrium:

$$p_1^* = \frac{3w + c_1 + c_2}{3} + \frac{c_1}{3}, \quad p_2^* = \frac{3w + c_1 + c_2}{3} + \frac{c_2}{3}.$$

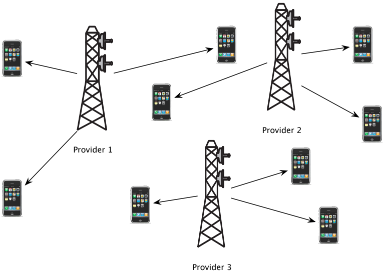
The Hotelling Model

- Illustration of Equilibrium
  - Geometrically, the Nash equilibrium is the intersection of both players' best response curves.



Section 6.3:
Wireless Service Provider Competition Revisited

Network Model



- A set $\mathcal{J} = \{1, \ldots, J\}$ of service providers
  - Provider j has a supply Q_j of resource (e.g., channel, time, power)
  - Providers operate on orthogonal spectrum bands
- A set $\mathcal{I} = \{1, \ldots, I\}$ of users
  - User i can obtain resources from multiple providers: $\boldsymbol{q}_i = (q_{ij}, \forall j \in \mathcal{J})$
  - User i's utility function is $u_i\left(\sum_{j=1}^{J} q_{ij} c_{ij}\right)$: increasing and strictly concave

An Example: TDMA

- Each provider j has a total spectrum band of W_j.
- q_{ij}: the fraction of time that user i transmits on provider j's band
  - Constraints: $\sum_i q_{ij} \leq 1$, for all $j \in \mathcal{J}$.
- c_{ij}: the data rate achieved by user i on provider j's band

$$c_{ij} = W_j \log(1 + \frac{P_i |h_{ij}|^2}{\sigma_{ij}^2 W_j})$$

  - P_i: user i's peak transmission power.
  - h_{ij}: the channel gain between user i and network j.
  - σ_{ij}^2: the Gaussian noise variance for the channel.
- $u_i \left(\sum_{j=1}^{J} q_{ij} c_{ij} \right)$: user i' utility of the total achieved data rate

Two-Stage Game

- Stage I: each provider $j \in \mathcal{J}$ announces a unit price p_j
  - Each provider i wants to maximize his revenue
  - Denote $\boldsymbol{p} = (p_j, \forall j \in \mathcal{J})$ as the price vectors of all providers.
- Stage II: each user $i \in \mathcal{I}$ chooses a demand vector $\boldsymbol{q}_i = (q_{ij}, \forall j \in \mathcal{J})$
  - Each user i wants to maximize his payoff (utility minus payment)
  - Denote $\boldsymbol{q} = (\boldsymbol{q}_i, \forall i \in \mathcal{I})$ as the demand vector of all users.
- Analysis based on backward induction

Goal: Derive the SPNE

- A price demand tuple $(\boldsymbol{p}^*, \boldsymbol{q}^*(\boldsymbol{p}^*))$ is a SPNE if no player has an incentive to deviate unilaterally at any stage of the game.
  - Each user i maximizes its payoff by choosing the optimal demand $\boldsymbol{q}_i^*(\boldsymbol{p}^*)$, given prices $\boldsymbol{p}^*$.
  - Each provider j maximizes its revenue by choosing price p_j^*, given other providers' prices $p_{-j}^* = (p_k^*, \forall k \neq j)$ and the user demands $\boldsymbol{q}^*(\boldsymbol{p}^*)$.

Stage II: User's Demand Optimization

- Each user $i \in \mathcal{I}$ solves a user payoff maximization (UPM) problem:

$$\textbf{UPM}: \max_{\boldsymbol{q}_i \geq \mathbf{0}} \left(u_i \left(\sum_{j=1}^{J} q_{ij} c_{ij} \right) - \sum_{j=1}^{J} p_j q_{ij} \right).$$

Stage II: User's Demand Optimization

- Each user $i \in \mathcal{I}$ solves a user payoff maximization (UPM) problem:

$$\textbf{UPM}: \max_{\boldsymbol{q}_i \geq \mathbf{0}} \left(u_i \left(\sum_{j=1}^{J} q_{ij} c_{ij} \right) - \sum_{j=1}^{J} p_j q_{ij} \right).$$

- Problem UPM may have more than one optimization solution $\boldsymbol{q}_i^*$
  - Since it is not strictly concave maximization problem in $\boldsymbol{q}_i^*$

Stage II: User's Demand Optimization

- Each user $i \in \mathcal{I}$ solves a user payoff maximization (UPM) problem:

$$\textbf{UPM}: \max_{\boldsymbol{q}_i \geq \mathbf{0}} \left(u_i \left(\sum_{j=1}^{J} q_{ij} c_{ij} \right) - \sum_{j=1}^{J} p_j q_{ij} \right).$$

- Problem UPM may have more than one optimization solution $\boldsymbol{q}_i^*$
  - Since it is not strictly concave maximization problem in $\boldsymbol{q}_i^*$
- Problem UPM has a unique solution of the effective resource x_i

Lemma (6.16)

- *For each user $i \in \mathcal{I}$, there exists a unique nonnegative value x_i^*, such that $\sum_{j \in \mathcal{J}} c_{ij} q_{ij}^* = x_i^*$ for every maximizer $\boldsymbol{q}_i^*$ of the UPM problem.*
- *For any provider j such that $q_{ij}^* > 0$, $p_j / c_{ij} = \min_{k \in \mathcal{J}} p_k / c_{ik}$.*

Decided vs. Undecided Users

Definition (Preference set)

For any price vector $\boldsymbol{p}$, user i's preference set is

$$\mathcal{J}_i(\boldsymbol{p}) = \left\{ j \in \mathcal{J} : \frac{p_j}{c_{ij}} = \min_{k \in \mathcal{J}} \frac{p_k}{c_{ik}} \right\}.$$

Decided vs. Undecided Users

Definition (Preference set)

For any price vector $\boldsymbol{p}$, user i's preference set is

$$\mathcal{J}_i(\boldsymbol{p}) = \left\{ j \in \mathcal{J} : \frac{p_j}{c_{ij}} = \min_{k \in \mathcal{J}} \frac{p_k}{c_{ik}} \right\}.$$

- A decided user has a singleton preference set.
- An undecided user has a preference set that includes more than one provider.

Decided vs. Undecided Users

Definition (Preference set)

For any price vector $\boldsymbol{p}$, user i's preference set is

$$\mathcal{J}_i(\boldsymbol{p}) = \left\{ j \in \mathcal{J} : \frac{p_j}{c_{ij}} = \min_{k \in \mathcal{J}} \frac{p_k}{c_{ik}} \right\}.$$

- A decided user has a singleton preference set.
- An undecided user has a preference set that includes more than one provider.
- One can use a bipartite graph representation (BGR) to uniquely determine the demands of undecided users.
- This will lead to all users' optimal demand $\boldsymbol{q}^*(\boldsymbol{p}) = (\boldsymbol{q}_i^*(\boldsymbol{p}), \forall i \in \mathcal{I})$ in Stage II.

Stage I: Provider's Revenue Optimization

- Each provider $j \in \mathcal{J}$ solves a provider revenue maximization (PRM) problem

$$\textbf{PRM}: \max_{p_j \geq 0} \quad p_j \cdot \min\left(Q_j, \sum_{i \in \mathcal{I}} q_{ij}^*(p_j, p_{-j})\right)$$

- Solving the PRM problem requires the consideration of other providers' prices p_{-j}.

Benchmark: Social Welfare Optimization (Ch. 4)

SWO: Social Welfare Optimization Problem

$$
\begin{aligned}
\text{maximize} \quad & \sum_{i \in \mathcal{I}} u_i\left(x_i\right) \\
\text{subject to} \quad & \sum_{j \in \mathcal{J}} q_{ij} c_{ij} = x_i, \ \forall i \in \mathcal{I}, \\
& \sum_{i \in \mathcal{I}} q_{ij} = Q_j, \ \forall j \in \mathcal{J}, \\
\text{variables} \quad & q_{ij}, x_i \geq 0, \ \forall i \in \mathcal{I}, j \in \mathcal{J}.
\end{aligned}
$$

Stage I: Provider's Revenue Optimization

Theorem

Under proper technical assumptions, the unique socially optimal demand vector $\boldsymbol{q}^*$ *and the associated Lagrangian multiplier vector* $\boldsymbol{p}^*$ *of the SWO problem constitute the unique SPNE of the provider competition game.*

Optimization, Game, and Algorithm

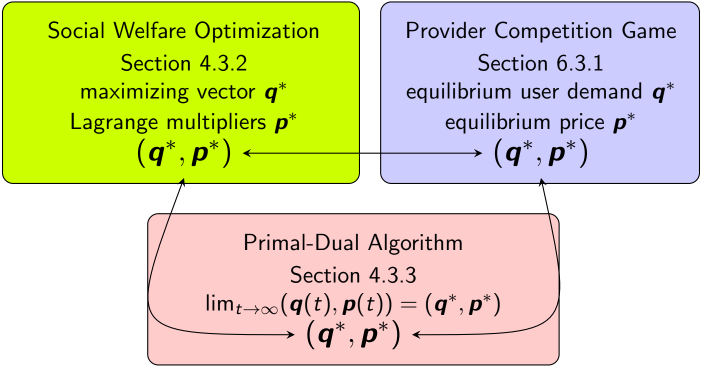
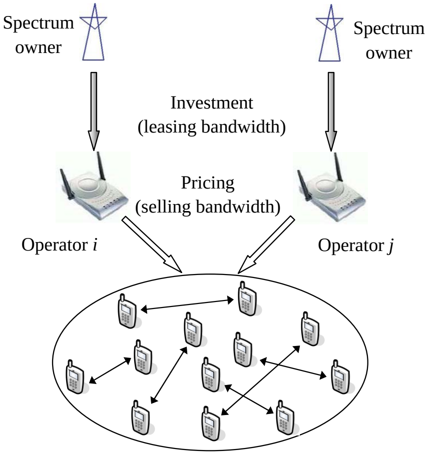


Figure: Relationship among different concepts

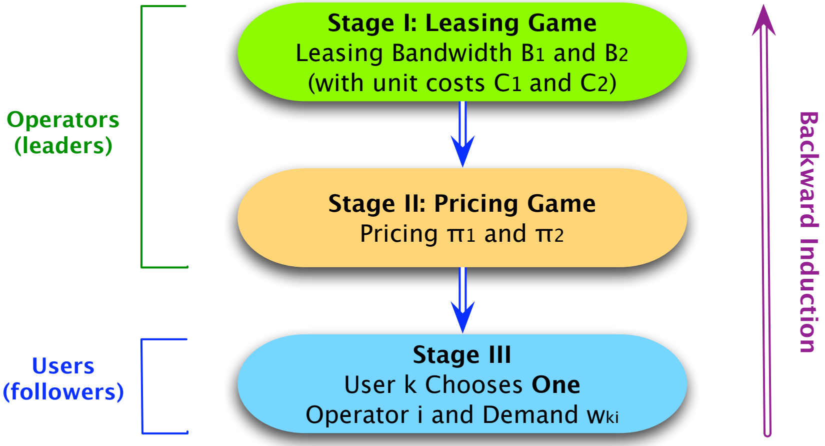
Section 6.4: Competition with Spectrum Leasing

Network Model



Secondary users (transmitter-receiver pairs)

Three-Stage Multi-leader-follower Game



Stage IIII: Users' Bandwidth Demands

- User k's payoff of choosing operator $i = 1, 2$

$$u_k(\pi_i, w_{ki}) = w_{ki} \ln \left(\frac{P_i^{\max} h_i}{n_0 w_{ki}} \right) - \pi_i w_{ki}$$

  - High SNR approximation of OFDMA system
  - Optimal demand: $w_{ki}^*(\pi_i) = \arg\max_{w_{ki} \geq 0} u_k(\pi_i, w_{ki}) = g_k e^{-(1+\pi_i)}$
  - Optimal payoff: $u_k(\pi_i, w_{ki}^*(\pi_i))$

- User k prefers the "better" operator: $i^* = \arg\max_{i=1,2} u_k(\pi_i, w_{ki}^*(\pi_i))$

- Users demands may not be satisfied due to limited resource
  - Difference between preferred demand and realized demand

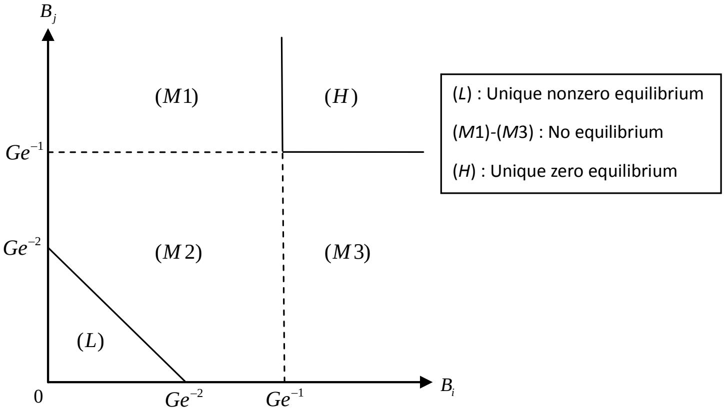
Stages II: Pricing Game

- Players: two operators
- Strategies: $\pi_i \geq 0, i = 1, 2$
- Payoffs: profit R_i for operator $i = 1, 2$:

$$R_i(B_i, B_j, \pi_i, \pi_j) = \pi_i Q_i(B_i, B_j, \pi_i, \pi_j) - B_i C_i$$

Stage II: Pricing Equilibrium

- Symmetric equilibrium: $\pi_1^* = \pi_2^*$.
- Threshold structure:
  - Unique positive equilibrium exists $B_1 + B_2 \leq Ge^{-2}$.



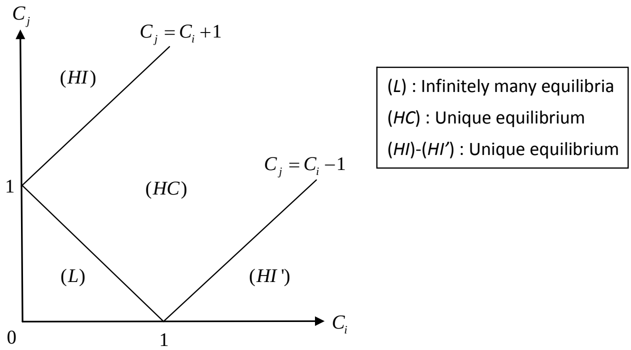
Stage I: Leasing Game

- Players: two operators
- Strategies: $B_i \in [0, \infty), i = 1, 2$, and $B_1 + B_2 \leq Ge^{-2}$.
- Payoffs: profit R_i for operator $i = 1, 2$:

$$R_i(B_i, B_j) = B_i \left(\ln \left(\frac{G}{B_i + B_j} \right) - 1 - C_i \right)$$

Stage I: Leasing Equilibrium

- Linear in wireless characteristics $G = \sum_i g_i$;
- Threshold structure:
  - Low costs: infinitely many equilibria
  - High comparable costs: unique equilibrium
  - High incomparable costs: unique monopoly equlibrium



Equilibrium Summary (Assuming $C_i \leq C_j$)

Costs	LOW $C_i + C_j \leq 1$	HC $C_i + C_j > 1$, $C_j - C_i \leq 1$	HI $C_j > 1 + C_i$
equilibria	Infinite	Unique	Unique
(B_i^*, B_j^*)	$(\rho G e^{-2}, (1-\rho) G e^{-2})$, $\rho \in [C_j, (1-C_i)]$	$\left(\frac{(1+C_j-C_i)G}{2e^{\frac{C_i+C_j+3}{2}}}, \frac{(1+C_i-C_j)G}{2e^{\frac{C_i+C_j+3}{2}}}\right)$	$(Ge^{-(2+C_i)}, 0)$
(π_i^*, π_j^*)	$(1,1)$	$\left(\frac{C_i+C_j+1}{2}, \frac{C_i+C_j+1}{2}\right)$	$(1+C_i, N/A)$
User SNR	e^2	$e^{\frac{C_i+C_j+3}{2}}$	e^{2+C_i}
User Payoff	$g_k e^{-2}$	$g_k e^{-\left(\frac{C_i+C_j+3}{2}\right)}$	$g_k e^{-(2+C_i)}$

- Users achieve the same SNR
- User k's payoff is linear in g_k

Robustness of Results

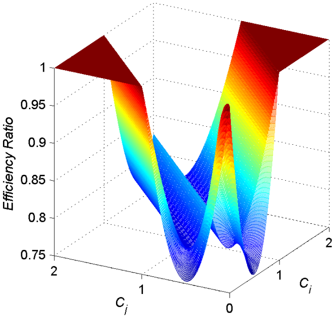
- To obtain closed form solutions, we have assumed
  - All users achieve high SNR
- Previous observations still hold in the general case
  - Users operate in general SNR regime: $r_{ki}(w_{ki}) = w_{ki} \ln\left(1 + \frac{P_k^{\max} h_k}{n_0 w_{ki}}\right)$

Impact of Duopoly Competition on Operators

- Benchmark: **Coordinated** Case
  - Operators cooperate in investment and pricing to maximize total profit
- Define

$$\text{Efficiency Ratio} = \frac{\text{Total Profit in Competition Case}}{\text{Total Profit in Coordinated Case}}$$

- Price of Anarchy $= \min_{C_i, C_j}$ Efficiency Ratio$= 0.75$.



Section 6.5: Chapter Summary

Key Concepts

- Theory: Game Theory
  - Dominant Strategy
  - Pure and Mixed Strategy Nash Equilibrium
  - Subgame Perfect Nash Equilibrium
- Theory: Oligopoly
  - Cournot competition
  - Bertrand competition
  - Hotelling competition
- Application: Wireless Network Competition Revisited
- Application: Competition with Spectrum Leasing

References and Extended Reading

- J. Huang, "How Do We Play Games?" online video tutorial, on YouKu (http://www.youku.com/playlist_show/id_19119535.html) and iTunesU (https://itunes.apple.com/hk/course/how-do-we-play-games/id642100914)
- V. Gajic, J. Huang, and B. Rimoldi, "Competition of Wireless Providers for Atomic Users," *IEEE Transactions on Networking*, vol. 22, no. 2, pp. 512 - 525, April 2014
- L. Duan, J. Huang, and B. Shou, "Duopoly Competition in Dynamic Spectrum Leasing and Pricing," *IEEE Transactions on Mobile Computing*, vol. 11, no. 11, pp. 1706 1719, November 2012

http://ncel.ie.cuhk.edu.hk/content/wireless-network-pricing